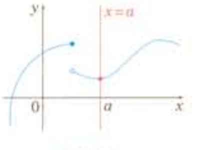
SPARKCHARTS™

PRE-CALCULUS

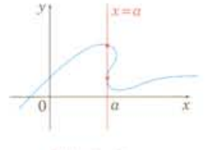
FUNCTIONS

DEFINITION

- A **relation** is a set of ordered pairs of values (x, y) that "go together." Plotted on the Cartesian plane, a relation is any set of points. **Ex:** The unit circle in the plane is a relation; it is the set of points (x, y) that satisfy $x^2 + y^2 = 1$.
- A **function** is a set of ordered pairs (x, y) so that for each x-value, there is no more than one y-value. Plotted on the Cartesian plane, a function must pass the **vertical line test:** Every vertical line cuts the graph of the function at most once.



Function



Not a function

- A function can be thought of as a rule for generating values. Plug in a value for the **independent variable** (frequently x) and receive a value for the **dependent variable** (frequently y). We say that "y is a function of x," and write $y = f(x)$—"y equals f of x".
- The **domain** of a function is the set of all allowable values that can be plugged in for the independent variable. **Ex:** The domain of the function $f(x) = \frac{1}{x}$ is all real numbers except 0.
- The **range** is the set of all possible outputs (values of the dependent variable). **Ex:** The range of the function $y = \sin x$ is the set of all real numbers between -1 and 1, inclusive (the closed interval $[-1, 1]$).

WRITING FUNCTIONS DOWN

- A **table** keeps track of input values (**Ex:** time of day) and corresponding output values (**Ex:** number of trucks on U.S. 66) of a function. There may not be a universal equation that describes a function.
- An **equation** such as $f(x) = x^2 + 1$ describes how to numerically manipulate the incoming variable (here, x) to get the output value $f(x)$.
- A **graph** represents a function visually. If $y = f(x)$, then plotting many points $(x, f(x))$ on the plane will give a picture of the function. Usually, the independent variable is represented horizontally, and the dependent variable vertically. Again, there need not be a single equation for a function described graphically, but the graph must pass the vertical line test.

VERY FAMILIAR FUNCTIONS

- **Linear Functions:** The equations whose graph is a line $(Ax + By = C)$ give functions for y in terms of x when they are converted to the form $y = mx + b$. *Exception: If $B = 0$, the line is vertical and the equation $x = \frac{C}{A}$ is not a function.*
- **Quadratic Functions:** The equations whose graph is a **parabola** $(y = ax^2 + bx + c)$ are quadratic functions.

EXPONENT AND LOGARITHM SUMMARY

For more on exponents and logarithms, see the Algebra I and Algebra II SparkCharts.

EXPONENT RULE:	LOGRITHM RULE:
$a^1 = a$	$\log_a a = 1$
$a^0 = 1$ (unless $a = 0$) The expression 0^0 is undefined	$\log_a 1 = 0$ for all (positive) bases a.
$a^{\log_a b} = b$	$\log_a a^n = n$
$a^{m+n} = a^m a^n$	$\log_a (bc) = \log_a b + \log_a c$
$a^{m-n} = \frac{a^m}{a^n}$	$\log_a \left(\frac{b}{c}\right) = \log_a b - \log_a c$
$a^{-n} = \frac{1}{a^n}$	$\log_a \frac{1}{b} = -\log_a b$
$a^{mn} = (a^m)^n$	$\log_a b^n = n \log_a b$
$a^{\frac{1}{n}} = \sqrt[n]{a}$	$\log_a \sqrt[n]{b} = \frac{1}{n} \log_a b$
$a^{\frac{m}{n}} = \sqrt[n]{a^m} = (\sqrt[n]{a})^m$	$\log_a \sqrt[n]{b^m} = \frac{m}{n} \log_a b$

REVIEW OF EXPONENTS AND LOGARITHMS

- **Exponents**: In the expression $a^n = b$, a is the **base**, n is the **exponent**.
- If n is an integer, then a^n represents repeated multiplication: $a^n = \underbrace{a \cdot a \cdots a}_{n \text{ times}}$, and b is called the n^{th} **power** of a.
- If n is any rational number (say, $n = \frac{r}{s}$), then $a^n = a^{\frac{r}{s}} = \sqrt[s]{a^r}$.
- **Logarithms**: $\log_a b = n$ is the power to which you raise a to get b. REMEMBER: Logarithms are exponents: $\log_a b = n$ if and only if $a^n = b$.

  Both a and b must be positive; also $a \neq 1$.
- If base $a > 1$ then $\log_a b > 0$ when $b > 1$ and $\log_a b < 0$ when $b < 1$.
- $\log b$ means $\log_{10} b$. It is often used in applied sciences and by calculators.
- ***e*** is a special irrational number (approximately 2.71828) often used as a base for logarithms. The logarithm base e is called the **natural logarithm** and is written $\log_e x = \ln x$. *The natural log follows all logarithm rules.*
  - Any logarithmic expression can be written in terms of natural logarithms using the change of base formula $\log_a b = \frac{\ln b}{\ln a}$.

ADDITIONAL EXPONENT RULES	
$(ab)^n = a^n b^n$	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$
$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$	

CHANGE OF BASE RULE FOR LOGARITHMS

Changing the base is multiplying by a constant.

$\log_a b = \log_a c \log_c b$. The c is "canceled."

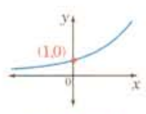
Also, $\log_a b = \frac{1}{\log_b a}$.

EXPONENTIAL AND LOGARITHMIC FUNCTIONS

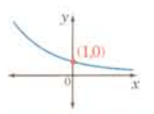
BASIC EXPONENTIAL FUNCTIONS: *f(x) = aˣ*

An **exponential function** has the basic equation $f(x) = a^x$. Here, a must be positive and $a \neq 1$.

- **Domain**: all real numbers. **Range**: all positive numbers. ***y*-intercept** at 1.
- **Behavior**: If base $a > 1$, the function is constantly increasing; it grows extremely fast for positive x, and approaches 0 for negative x. If $a < 1$, the function is constantly decreasing; it takes very large values for negative x and tends towards 0 for positive x.

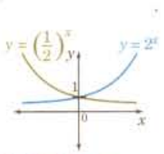


$y = a^x$ with $a > 1$

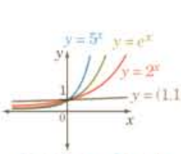


$y = a^x$ with $0 < a < 1$

- The graph of $f(x) = \left(\frac{1}{a}\right)^x$ is a reflection of the graph of $f(x) = a^x$ over the y-axis. See *Reflections over the Axes*.
- For $a > 1$, the graph of $f(x) = a^x$ grows faster the larger a is.



Reciprocal bases

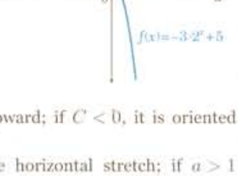


Bases greater than 1

$f(x) = e^x \approx (2.718)^x$ is often thought of as the quintessential exponential function. Any exponential function can be reexpressed with base e: if $f(x) = a^x$, then, since $a = e^{\ln a}$, we have $f(x) = e^{x \ln a}$. If $a > 1$, then the graph of $f(x) = a^x$ is the graph of $f(x) = e^x$ stretched in the x-direction by a factor of $\ln a$. Every exponential function has the same basic shape.

GENERAL EXPONENTIAL FUNCTIONS

The most general exponential function is given by the equation $f(x) = Ca^x + K$. Equivalently, we let $b = \ln a$ and write $f(x) = Ce^{bx} + K$. (Note that C can swallow any constant added to x, since $a^{x+h} = (a^h)a^x$.)



- $|C|$ determines the vertical stretch of the graph. Stretching the graph vertically has the same effect as shifting the graph horizontally. If $C > 0$, the graph is oriented upward; if $C < 0$, it is oriented downward.
- a (or b) determines the horizontal stretch; if $a > 1$ $(b > 0)$, the graph increases to the right; if $0 < a < 1$ $(b < 0)$, it increases to the left.
- K is the value the function approaches in the exponential decay. The line $y = K$ is a **horizontal asymptote**. The ***y*-intercept** is $C + K$.

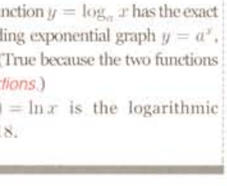
FINDING AN EQUATION FOR AN EXPONENTIAL FUNCTION FROM THE GRAPH

Two points and the height of the asymptote are sufficient to find the equation of an exponential graph.

- If we know asymptote $y = K$, y-intercept y_0, and point (x_1, y_1): The function is $y = Ca^x + K$, where $C = y_0 - K$ and a is the base such that $a^{x_1} = \frac{y_1 - K}{C}$.
- If we know asymptote $y = K$ and two points (x_0, y_0) and (x_1, y_1): Divide the two equations $y_0 - K = Ca^{x_0}$ and $y_1 - K = Ca^{x_1}$ to find base a such that $a^{x_1 - x_0} = \frac{y_1 - K}{y_0 - K}$. Use a to find $C = \frac{y_1 - K}{a^{x_1}}$.

LOGARITHMIC FUNCTIONS

A logarithmic function has the form $y = \log_a x$. The domain is positive numbers only ($\log_a 0$ is undefined); the range is all real numbers. There is a vertical asymptote at $x = 0$. The graph is always increasing; it grows very quickly for $0 < x < 1$, crosses the x-axis at $x = 0$, and then continues growing extremely slowly—slower than any root function—for $x > 1$.



- The graph of the logarithmic function $y = \log_a x$ has the exact same shape as the corresponding exponential graph $y = a^x$, reflected over the line $y = x$. (True because the two functions are inverses. See *Inverse Functions*.)
- **Natural logarithm:** $f(x) = \ln x$ is the logarithmic function with base $e \approx 2.718$.

SPARKCHARTS™

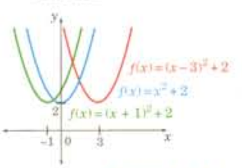
PRE-CALCULUS

CHANGING A FUNCTION: SHIFTS, STRETCHES, REFLECTIONS

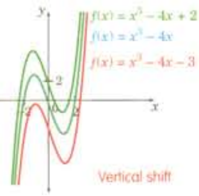
TRANSLATIONS

A **translation** of a function is a **shift** vertically, horizontally, or both; the shape, the orientation, and the scale of the graph are all unchanged.

- **Vertical translation:** Adding a constant c to the equation will shift the function vertically c units (up if c is positive, down if c is negative). The new function $y = f(x) + c$ has the same shape and the same domain as the original function.
- **Horizontal translation:** The function $y = f(x - c)$ is a shift of the original function c units horizontally (to the right if c is positive, left if c is negative). The new function has the same shape and the same range as the original function.



Horizontal shift

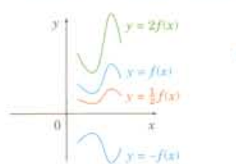


Vertical shift

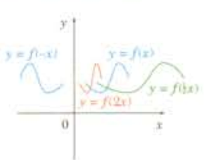
STRETCHES

The graph of a function can be **stretched** or **compressed,** horizontally or vertically (or both), by multiplying by a constant.

- **Vertical stretching, compressing:** For positive c, the function $y = cf(x)$ is a vertical stretch or compression of the original function. If $c > 1$, then the function $y = cf(x)$ is stretch by a factor of c. If $c < 1$, then $y = cf(x)$ is a compression by a factor of c. Horizontal distances remain unchanged.
- **Horizontal stretching, compressing:** Again, for positive c, the function $y = f\left(\frac{x}{c}\right)$ is a horizontal stretch of the original function if $c < 1$ (a compression if $c > 1$) by a factor of c. Vertical distances remain the same.



Vertical stretch

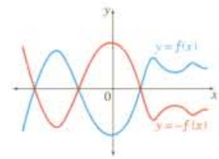


Horizontal stretch

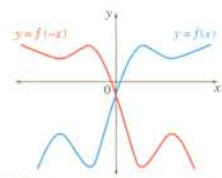
REFLECTIONS OVER THE AXES

Reflecting a function over the axes creates a new function which is the same shape and size as the original.

- **Reflection over the *x*-axis:** The function $y = -f(x)$ is a reflection of the original function over the x-axis. The new function has the same domain as the original; the range is the negative of the original range.
- **Reflection over the *y*-axis:** The function $y = f(-x)$ is a reflection of the original function over the y-axis. The new function has the same range as the original; the domain is the negative of the original domain.

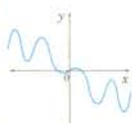


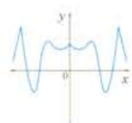
Reflecting $y = f(x)$ over x-axis



Reflecting $y = f(x)$ over y-axis

- If $f(x) = f(-x)$, then $f(x)$ is called **even;** it remains unchanged when reflected over the y-axis. **Ex:** $\cos x$ is an even function.
- If $f(x) = -f(-x)$, then $f(x)$ is called **odd.** A reflection over the x-axis is the same as a reflection over the y-axis. Equivalently, a $180°$ rotation of $f(x)$ around the origin leaves $f(x)$ unchanged. **Ex:** $\sin x$ is an odd function.
- **Reflection over the line $y = x$:** Switch the roles of x and y in the equation; the resulting relation (set of points in the plane) is a reflection over the line $y = x$. If you can solve the new expression for y, the reflected relation is a function—the inverse function. See below.


Odd function


Even function

INVERSE FUNCTIONS

If the function $f(x)$ passes the "horizontal line test" in its domain—$f(x)$ never takes the same value twice—then $f(x)$ has a unique **inverse** $f^{-1}(x)$ whose domain is the range of $f(x)$ and *vice versa.*

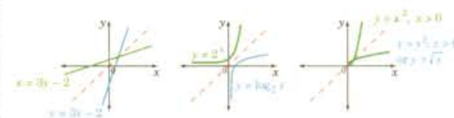
- To find the inverse function, switch the roles of x and y in the equation, effectively writing $x = f(y)$. Then solve for y. If you can solve for y "reversibly," then the function has an inverse.
- **Ex:** Linear function: $y = mx + b$. The inverse function is $y = \frac{1}{m}(x - b)$.
- **Ex:** Exponential function $y = a^x$. The inverse function is $y = \log_a x$.

NOTE: If $f(x)$ takes the same value more than once, we restrict the domain before taking the inverse. **Ex:** $y = x^2$ on the whole real line has no inverse, but the function $y = x^2$ on the positive reals only has the inverse $y = \sqrt{x}$.

- Graphically, $y = f^{-1}(x)$ has the same shape as the orginal function, but is reflected over the slanted line $y = x$. **Ex:** $y = 2^x$ and $y = \log_2 x$ are inverse functions. See graphs below.

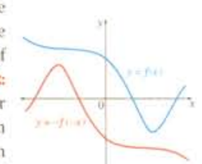
Properties of the inverse function

- It is a two-sided inverse: $f^{-1}(f(x)) = x$ for all x in the domain of $f(x)$ and $f(f^{-1}(x)) = x$ for all x in the domain of $f^{-1}(x)$.
- The inverse of the inverse function is the original function: $(f^{-1})^{-1}(x) = f(x)$.


$f(x) = 3x - 2$ $f(x) = 2^x$ $x \geq 0$

ROTATIONS

- **Rotating 180°:** A rotation of $180°$ is the same thing as a flip over the x-axis followed by a flip over the y-axis (or *vice versa,* though, in general, order of flips matters). Thus $y = -f(-x)$ is the equation of a function whose graph is a half-circle rotation of the original. **Odd** functions (**Ex:** $\sin x$, x^3) are unchanged after such a rotation. The domain and range of the new function are the negatives of the original function's domain and range.

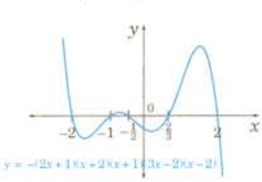


GENERAL POLYNOMIAL FUNCTIONS

For more background information on polynomials, see the SparkChart on Algebra II.

POLYNOMIAL REVIEW

A general polynomial in one variable can be reduced to the form $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. The constants $a_0, a_1, \ldots, a_n$ are the **coefficients**; expressions connected by $\pm$ signs are called **terms.** Two terms are **"like" terms** if they involve the same power of x; like terms can be collected and added together to simplify the polynomial. The **degree** of the polynomial is the highest power of x of any term after the polynomial is simplified; that term is called the **leading term**, and its coefficient is the **leading coefficient**. The term that involves no xs is the **constant term**. **Ex:** The polynomial above has degree n, leading term $a_n x^n$, leading coefficient a_n, and constant term a_0.



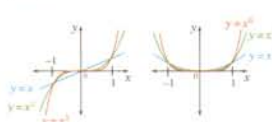
Polynomial of degree 5
Roots: $-2, -1, -\frac{1}{2}, \frac{2}{3}, 2$
4 "turns"

- A **root** (or a **zero**) of a polynomial is any number a such that $f(a) = 0$. On a graph, this corresponds to crossing the x-axis.
- The domain of any polynomial function is all real numbers. A graph is always "smooth"—no kinks.
- A polynomial of degree n will have no more than $n - 1$ "turns"—changes of direction—in the graph; it will cross the x-axis no more than n times (and so have at most n roots).

SIMPLEST POLYNOMIAL FUNCTIONS $f(x) = x^n$

The polynomial functions $f(x) = x^n$ come in two overall shapes.

- If n is odd, $f(x) = x^n$ goes to $-\infty$ for negative x and $+\infty$ for positive x. The range is all real numbers. The function crosses the x-axis at $x = 0$.
- If n is even, $f(x) = x^n$ goes off to $+\infty$ for large $|x|$ both positive and negative. The function is always nonnegative; it touches the x-axis at $x = 0$.


$y = x^n$ for odd n, $y = x^n$ for even n.

As n increases, $f(x) = x^n$ becomes flatter near the origin and steeper everywhere else for both odd and even n.

LOOKING FOR ROOTS—THEOREMS

The search for roots plays a big role in polynomial life. Factoring is the way to go.

- **Factor Theorem:** If a is a root of the polynomial $f(x)$, then we can express $f(x) = (x - a)g(x)$ for some other polynomial $g(x)$. In other words, a is a root if and only if $x - a$ is a factor of $f(x)$. To use:
  1. Every time you find a root a, factor out $x - a$ from the polynomial and continue the hunt for roots on the quotient.
  2. Whenever a polynomial has a linear factor $ax + b$, then $-\frac{b}{a}$ is a root.
- **Rational Roots Theorem:** If the polynomial with leading coefficient a and constant term b has a rational root, then the root is in the form $\pm\frac{r}{s}$, where r is a factor of b, and s is a factor of a.
  - To check for rational roots, list the factors s of the leading coefficient and the factors r of the constant term. Make all the possible fractions $\pm\frac{r}{s}$ and plug them in to the polynomial to check if they are roots.

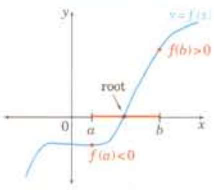
GENERAL POLYNOMIAL BEHAVIOR FOR LARGE $|x|$

For any polynomial function of degree n, as x gets very large, either positively or negatively, the leading term will dominate and determine the behavior of the function.

	n odd		*n* even	
Leading coefficient > 0		At least 1 root. Range is all real numbers.		A minimum value for the function exists. May or may not have roots.
Leading coefficient < 0		At least 1 root. Range is all real numbers.		A maximum value for the function exists. May or may not have roots.

GENERAL POLYNOMIAL FUNCTIONS (CONTINUED)

- Real roots: Harder to find. **Intermediate Value Theorem for Polynomials:** If $f(x)$ is a polynomial, and for some two numbers a and b, we have $f(a) > 0$ and $f(b) < 0$ (or *vice versa*), then the polynomial $f(x)$ has a root between a and b. This is intuitive if we believe that polynomial functions always have smooth graphs.



Intermediate Value Theorem for Polynomials

- **Descartes' Rule of Signs:** The number of positive real roots of a polynomial $f(x)$ is equal to or an even number less than the number of "sign reversals" in $f(x)$.

Ex: The polynomial $3x^5 - x^4 + 5x^3 + 7x^2 - 2x + 5$ has 4 sign reversals, so it has 4, 2, or 0 positive roots.

- Also, the number of negative roots of $f(x)$ is equal to or an even number less than the number of sign reversals in $f(-x)$. **Ex:** With $f(x)$ as above, $f(-x) = -3x^5 - x^4 - 5x^3 + 7x^2 + 2x + 5$. Since there is 1 sign reversal, $f(x)$ must have exactly 1 negative root.

SKETCHING A GENERAL POLYNOMIAL WITHOUT A CALCULATOR

1. Determine the behavior of the polynomial for large $|x|$.
2. Find all the roots you can:
   a. Factor the polynomial as much as possible to find roots and reduce it to terms of smaller degree.
   b. Use the Rational Roots Theorem on the unfactored pieces to find all rational roots. For each root a, divide out $x - a$ to reduce the degree.
   c. Use Descartes' Rule of Signs or the Intermediate Value Theorem to estimate number and location of real roots.
3. Plot all the real roots. For each interval between the roots, test a point to see if the graph is positive or negative on the interval. (A polynomial will cross (as opposed to touch) the x-axis at a root if and only if its multiplicity is odd.)
4. Sketch the curve.

RATIONAL FUNCTIONS

A **rational function** is a quotient of two polynomials: $f(x) = \frac{p(x)}{q(x)}$, where $q(x)$ is not the zero polynomial. The **domain** of the function is all real numbers except the roots of $q(x)$.

- An **asymptote** is a line, often vertical or horizontal, that a function gets very close to—but never quite touches—as $x \to \infty$ or $x \to -\infty$ (often both). Rational functions will more often than not have at least one vertical asymptote.
- On a graph, an asymptote will usually be marked as a dashed line.

ZEROES

A rational function $\frac{p(x)}{q(x)}$ will cross the x-axis at all the roots of $p(x)$ that are not also roots of $q(x)$.

- More precisely, $\frac{p(x)}{q(x)}$ will also have a zero at a if it is a root of both $p(x)$ and $q(x)$, but the multiplicity of a as a root of $p(x)$ is greater than the multiplicity of a as a root of $q(x)$.

CALCULUS NOTATION

This notation is frequently used to describe the "end behavior" of a function (i.e., what happens when $|x|$ approaches ∞) or to describe the function near points where it is not defined (such as vertical asymptotes).

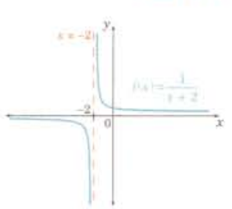
Usage: Ex: If $f(x) = x^n$, then $f(x) \to \infty$ as $x \to \infty$. If n is odd, then $f(x) \to -\infty$ as $x \to -\infty$. If n is even, $f(x) \to +\infty$ as $|x| \to \infty$.

Notation	Meaning
$x \to \infty$	x increases without bound
$x \to -\infty$	x decreases without bound
$\lvert x\rvert \to \infty$	x increases both positively and negatively
$x \to a$	x approaches a
$x \to a^+$	x gets close to a while staying greater than a; x approaches a from the right
$x \to a^-$	x gets close to a while staying less than a; x approaches a from the left

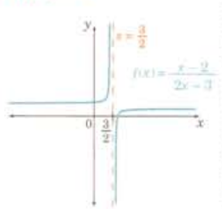
VERTICAL ASYMPTOTES

Function $f(x)$ has a vertical asymptote given by the equation $x = a$ when the value of of the function increases without bound as x approaches a.

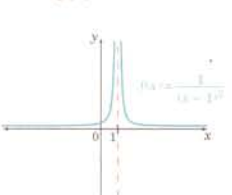
Four types of rational function behavior near a vertical asymptote:



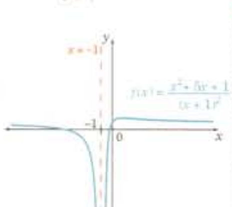
Near a, $f(x) < 0$ for $x < a$ and $f(x) > 0$ for $x > a$.



Near a, $f(x) > 0$ for $x < a$ and $f(x) < 0$ for $x > a$.



Near a, $f(x) > 0$.



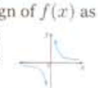
Near a, $f(x) < 0$.

- In other words, $x = a$ is a vertical asymptote if $f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a^-$ or $x \to a^+$. For rational functions, $f(x) \to \pm\infty$ as $x \to a$ from both sides.

A rational function $\frac{q(x)}{p(x)}$ will have a vertical asymptote at every root of $q(x)$ that is *not* also a root of $p(x)$.

- More precisely, $\frac{p(x)}{q(x)}$ will also have vertical asymptote $x = a$ if a is a root of both $p(x)$ and $q(x)$, but the multiplicity of a as a root of $q(x)$ is greater than the multiplicity of a as a root of $p(x)$.

- Determining behavior of $f(x)$ near vertical asymptote $x = a$: check the sign of $f(x)$ (no need to compute values) as $x \to a^-$ and $x \to a^+$. Easiest to do when both numerator and denominator are completely factored.
- **Ex:** The function $f(x) = \frac{(2x-1)(x+3)}{x}$ has vertical asymptote $x = 0$. When x approaches 0 from the left, $2x - 1 < 0$, $x + 3 > 0$ and $x < 0$. So the sign of $f(x)$ as $x \to 0^-$ is $\frac{(-)(+)}{(-)} = +$. The sign of $f(x)$ as $x \to 0^+$ is $\frac{(-)(+)}{(+)} = -$. Near 0, the function looks like the figure at right.



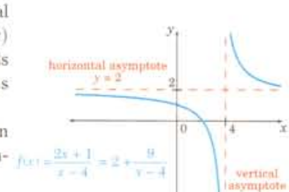
HORIZONTAL ASYMPTOTES

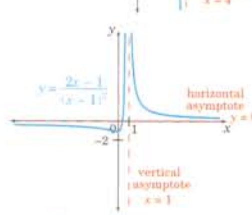
Function $f(x)$ has a **horizontal asymptote** at b if $f(x)$ approaches—but never reaches—the line $y = b$ for large $|x|$.

- More precisely, $y = b$ is a horizontal asymptote to $f(x)$ if $f(x) \to b$ as $x \to \infty$ or $x \to -\infty$. For rational functions, $f(x) \to b$ as $x \to \pm\infty$ on both sides.

If $\frac{p(x)}{q(x)}$ is a rational function with $p(x)$ and $q(x)$ polynomials with leading terms ax^n and bx^m, then:

- If $n < m$, then $y = 0$ is a horizontal asymptote.
- If $n = m$, then $y = \frac{a}{b}$ is a horizontal asymptote.
- If $n > m$, then there are no horizontal asymptotes. As $x \to \pm\infty$ on both sides, the function behaves more and more like the polynomial $\frac{a}{b}x^{n-m}$.



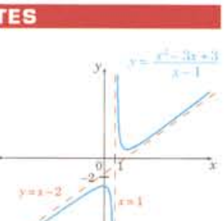


$f(x) = \frac{2x-1}{(x-1)^2}$ crosses its asymptote $y = 0$

- Rational functions may approach their horizontal asymptotes from above or from below (or from both above and below).
- Even though a function with horizontal asymptote $y = b$ will approach but never reach b for large $|x|$, the function may cross the line $y = b$ before it reaches its "asymptotic behavior" stage.

OBLIQUE ASYMPTOTES

If the degree of $p(x)$ is exactly one more than the degree of $q(x)$, then the rational function $\frac{p(x)}{q(x)}$ will have an **oblique** (a.k.a. **slanted** or **skew**) **asymptote.**

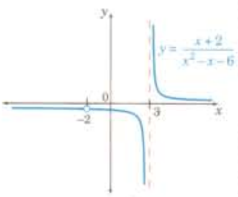


- To find the equation of a skew asymptote, use long division to express $\frac{p(x)}{q(x)} = ax + b + \frac{r(x)}{q(x)}$, where the degree of $r(x)$ is less than the degree of $q(x)$. The line $y = ax + b$ is a skew asymptote for the function.

HOLES ("REMOVABLE DISCONTINUITIES")

If vertical asymptotes disrupt the "smoothness" of a graph in a drastic way, **holes** (technically, "**removable discontinuities**") are gaps where a function *could* have been (but wasn't) defined smoothly.

- In the rational function $f(x) = \frac{p(x)}{q(x)}$, if a is a root of both $p(x)$ and $q(x)$ (with the same multiplicity), then—even though $f(a)$ is not defined because denominator $q(a) = 0$—the function passes over the point a without major hitches, leaving a small hole.



$f(x) = \frac{x+2}{(x+2)(x-3)}$ has a hole at $(-2, -\frac{1}{5})$

- Note: The function $f(x) = \frac{x+2}{(x+2)(x-3)}$ has all the same values as $g(x) = \frac{1}{x-3}$ except at $x = -2$: $f(-2)$ is undefined, while $g(-2) = -\frac{1}{5}$.

SUMMARY: RATIONAL FUNCTION SKETCHING

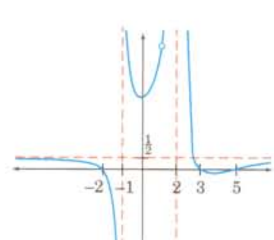
Suppose $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0}$:

Local behavior:

- If $p(x)$ and $q(x)$ have no roots in common:
  - $f(x)$ will cross the x-axis at each root of $p(x)$.
  - $f(x)$ will have a vertical asymptote at each root of $q(x)$.
- If $p(x)$ and $q(x)$ have a common root a, r is the multiplicity of a as a root of $p(x)$, and s the multiplicity of a as a root of $q(x)$:
  - If $r > s$, then $f(x)$ crosses the x-axis at a.
  - If $r = s$, then $f(x)$ has a hole at a.
  - If $r < s$, then $x = a$ is a vertical asymptote.

End behavior:

- If $n \le m$, then the function has a horizontal asymptote.
- If $n = m + 1$, then the function has an oblique asymptote.
- If $n > m + 1$, then the function approaches the graph of $\frac{a_n}{b_m}x^{m-n}$ asymptotically.



$y = \frac{(x+2)(x-3)(x-5)(x-1)}{2(x+1)(x-2)^2(x-1)}$

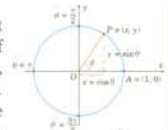
Horizontal asymptote: $y = \frac{1}{2}$
Vertical asymptote: $x = -1$ and $x = 2$
Removable discontinuity at $x = 1$

TRIG SUMMARY

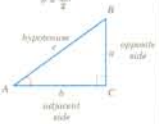
TRIGONOMETRIC FUNCTIONS

Trigonometric functions are commonly thought of in two ways:

1. Any angle θ defines a point $P = (x, y)$ on the **unit circle**: if O is the origin and $A = (1, 0)$, then P has $m\angle AOP = \theta$. Trigonometric functions are defined in terms of x and y, the coordinates of point P.



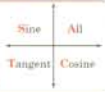
2. Trigonometric functions are also given by ratios of side lengths of a right triangle with acute angles θ and $\frac{\pi}{2} - \theta$. For $\theta > \frac{\pi}{2}$, apply the right triangle definitions to a **reference angle** (if $\frac{\pi}{2} < \theta < \pi$, $\theta_{\text{ref}} = \pi - \theta$; if $\pi < \theta < \frac{3\pi}{2}$, $\theta_{\text{ref}} = \theta - \pi$; etc.), and attach the appropriate $\pm$ sign (or just use the unit circle).



Func.	Unit circle	Right triangle	Domain	Range
$\sin\theta$	y	$\frac{\text{opp}}{\text{hyp}}$	all real numbers	$[-1, 1]$
$\cos\theta$	x	$\frac{\text{adj}}{\text{hyp}}$	all real numbers	$[-1, 1]$
$\tan\theta$	$\frac{y}{x}$	$\frac{\text{opp}}{\text{adj}}$	all reals except $k\pi + \frac{\pi}{2}$	all real numbers
$\csc\theta$	$\frac{1}{y}$	$\frac{\text{hyp}}{\text{opp}}$	all reals except $k\pi$	$(-\infty, -1] \cup [1, +\infty)$
$\sec\theta$	$\frac{1}{x}$	$\frac{\text{hyp}}{\text{adj}}$	all reals except $k\pi + \frac{\pi}{2}$	$(-\infty, -1] \cup [1, +\infty)$
$\cot\theta$	$\frac{x}{y}$	$\frac{\text{adj}}{\text{opp}}$	all reals except $k\pi$	all real numbers

SOHCAHTOA: "**S**ine is **O**pposite over **H**ypothenuse; **C**osine is **A**djacent over **H**ypothenuse; **T**angent is **O**pposite over **A**djacent."

All **S**tudents **T**ake **C**alculus tells which of the main trig functions are positive in which quadrants: I: **A**ll; II: **S**ine only; III: **T**angent only; IV: **C**osine only.



All trigonometric functions are **periodic** with **period** 2π (sin, cos, sec, csc) or π (tan, cot).

TRIGONOMETRIC IDENTITIES

Sum and difference formulas

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

Double-angle formulas

$\sin(2A) = 2 \sin A \cos A$

$\cos(2A) = \cos^2 A - \sin^2 A$
$= 2\cos^2 A - 1 = 1 - 2\sin^2 A$

Half-angle formulas

$\sin\frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$ $\quad\cos\frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$

Pythagorean identities

$\sin^2 A + \cos^2 A = 1$

$\tan^2 A + 1 = \sec^2 A \qquad 1 + \cot^2 A = \csc^2 A$

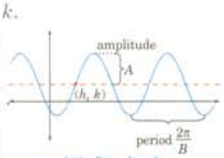
Special trigonometric values

θ (deg)	θ (rad)	$\sin\theta$	$\cos\theta$	$\tan\theta$
$0°$	0	$\frac{\sqrt{0}}{2} = 0$	1	0
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	$\frac{\sqrt{4}}{2} = 1$	0	undefined

GRAPHING SINE AND COSINE CURVES

Sinusoidal functions can be written in the form $y = A \sin B(x - h) + k$.

- $|A|$ is the **amplitude**.
- k is the is the **average value**: halfway between the maximum and the minimum value of the function.

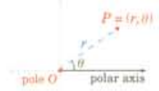


- $\frac{2\pi}{B}$ is the **period**. There are B cycles in every interval of length 2π, so $\frac{B}{2\pi}$ is the **frequency**.
- h is **phase shift**, or how far the beginning of the cycle is from the y-axis.

POLAR COORDINATES

Polar coordinates describe a point $P = (r, \theta)$ on a plane in terms of its distance r from the **pole**—usually, the origin O—and the (counterclockwise) angle θ that the line $\overline{OP}$ makes with the **polar axis**—usually, the positive x-axis. To identify a point, it is standard to limit $r \geq 0$ and $0 \leq \theta < 2\pi$, although

- $(-r, \theta) = (r, \theta \pm \pi)$, and
- $(r, \theta) = (r, \theta + 2n\pi)$ for integer n.



In Cartesian coordinates, $P = (r\cos\theta, r\sin\theta)$

CARTESIAN—POLAR CONVERSION

- **From Cartesian to polar:** $r = \sqrt{x^2 + y^2}$; $\theta = \tan^{-1}\frac{y}{x}$
- **From polar to Cartesian:** $x = r\cos\theta$; $y = r\sin\theta$

FUNCTIONS IN POLAR COORDINATES

Functions in polar coordinates usually define r in terms of θ. They need not (and almost never will) pass the vertical line test.

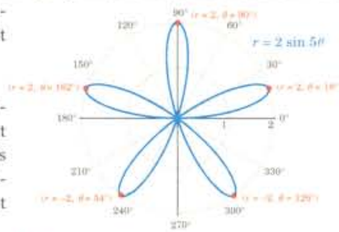
Circles:

- The graph of $r = a$ is a circle of radius $|a|$ centered at the origin.
- The graphs of equations $r = a\sin\theta$ and $r = a\cos\theta$ are circles of has radius $\left|\frac{a}{2}\right|$ centered at the (Cartesian coordinate) points $(0, \frac{a}{2})$ and $(\frac{a}{2}, 0)$, respectively.

Roses:

The graphs of equations $r = \sin n\theta$ and $r = \cos n\theta$ give roses with n petals if n is odd and $2n$ petals if n is even.

- **Cosine roses:** Always symmetric about the x-axis. If n is even, also symmetric about the y-axis.
- **Sine roses:** Always symmetric about the y-axis. If n is even, also symmetric about the x-axis.

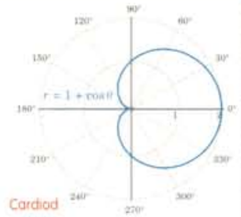


Limaçons and Cardioids:

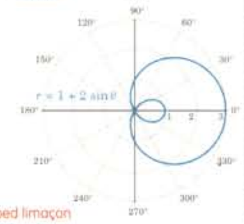
The graphs of equation $r = 1 \pm c\cos\theta$ and $r = 1 \pm c\sin\theta$ are called **limaçons**. When $c = 1$, the limaçon is called a **cardiod** (it is "heart"-shaped).

Assume that c is positive.

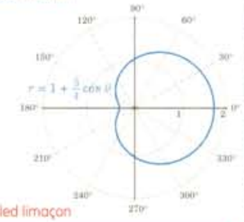
- If $c > 1$, the limaçon has an inner **loop**. If $\frac{1}{2} < c \leq 1$, the limaçon has a **dimple** (or **dent**). If $c \leq \frac{1}{2}$, the limaçon is convex (like a "squashed" circle).
- A **sine limaçon** is oriented up–down. The loop is on the bottom in $r = 1 + c\sin\theta$; on top in $r = 1 - c\sin\theta$.
- A **cosine limaçon** is oriented left-right. The loop is on the left in $r = 1 + c\cos\theta$; right in $r = 1 - c\cos\theta$.
- The graphs of $r = a \pm b\sin\theta$ and $r = a \pm b\cos\theta$ are limaçons stretched by a factor of $|a|$. Factor out a to get $c = \frac{b}{a}$. If a is negative, its orientation is reversed.



Cardiod



Looped limaçon



Dimpled limaçon

SYMMETRY

These tests *guarantee* symmetry, but they are not exhaustive.

- ***x*-axis symmetry:** If the equation is unchanged when θ is replaced by $-\theta$, the graph is symmetric about the x-axis.
- ***y*-axis symmetry:** If the equation is unchanged when θ is replaced by $\pi - \theta$, the graph is symmetric about the y-axis.
- **Origin symmetry:** If the equation is unchanged when r is replaced by $-r$, the graph is symmetric about the origin: the graph is unchanged when it is rotated 180°.
- The graph of the function $r = f(\theta - \alpha)$ is a rotation of the graph of $r = f(\theta)$ by α counterclockwise.
- The graph of the function $r = af(\theta)$ is a dilation of the graph of $r = f(\theta)$ by a factor of $|a|$. If a is negative, the graph is also reflected through the origin (same as a 180° rotation).

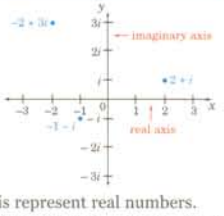
COMPLEX NUMBERS

COMPLEX NUMBERS

- **Imaginary numbers** are square roots of negative numbers. They are expressed as real multiples of i $(= \sqrt{-1})$.
- **Complex numbers** are all numbers $a + bi$ where a and b are real. Complex numbers are all sums and products of real and imaginary numbers.
  - The **complex conjugate** of $a + bi$ is $\overline{a + bi} = a - bi$. Also, $\overline{a - bi} = \overline{\overline{a + bi}} = a + bi$.
  - The product of a complex number and its conjugate is a real number: $(a + bi)(a - bi) = a^2 + b^2$.
- **Addition, subtraction,** and **multiplication:** Complex numbers are added and multiplied like polynomials, keeping the real and the imaginary part separate:
  $(a + bi) \pm (c + di) = (a + c) \pm (b + d)i$.
  For multiplication, use $i \cdot i = -1$:
  $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.
- **Division:** To divide one complex number by another, multiply top and bottom of the fraction by the conjugate of the denominator and simplify the numerator.
  $\frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$

COMPLEX PLANE

- Complex numbers can be represented as points on a plane (just like real numbers can be represented as points on a line). The number $a + bi$ is represented as the point (a, b).



- The horizontal axis is the **real axis**. Points on the x-axis represent real numbers.
- The vertical axis is the **imaginary axis**. Points on the y-axis represent imaginary numbers.
- The complex conjugate of a number is represented by the point reflected across the x-axis.
- The product of a number and its conjugate is the square of its distance from the origin: $(a + bi)(a - bi) = a^2 + b^2$.

TRIGONOMETRIC FORM: $r\cos q + i\sin q$

Trigonometric or **polar form** of a complex number comes from identifying the points on the complex plane with polar coordinates. Multiplication and division are simple in this form.

- In trigonometric form, $x + yi = r(\cos\theta + i\sin\theta)$. Here, $r = \sqrt{x^2 + y^2}$ is the **modulus**, or the distance of the point from the origin, and $\theta = \arctan\frac{y}{x}$ is the **argument**, or the angle that the line $\overline{OP}$ makes with the positive x-axis.
- Sometimes $\cos\theta + i\sin\theta$ is abbreviated as $\text{cis}\,\theta$ and this notation is called **"cis notation."**

PRODUCTS, QUOTIENTS, AND DEMOIVRE'S THEOREM

- **Multiplication:**
  $\big(r_1(\cos\theta_1 + i\sin\theta_2)\big)\big(r_2(\cos\theta_2 + i\sin\theta_2)\big)$
  $= r_1 r_2\big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big)$
  In cis notation, $(r_1\,\text{cis}\,\theta_1)(r_2\,\text{cis}\,\theta_2) = r_1 r_2\,\text{cis}(\theta_1 + \theta_2)$.
- **Division:**
  $\frac{r_1(\cos\theta_1 + i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}\big(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\big)$
  In cis notation, $\frac{r_1\,\text{cis}\,\theta_1}{r_2\,\text{cis}\,\theta_2} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2)$.
- **DeMoivre's Theorem—raising to powers:**
  $\big(r(\cos\theta + i\sin\theta)\big)^n = r^n(\cos n\theta + i\sin n\theta)$
  In cis notation, $(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,n\theta$.
- **Extracting roots:** The complex number $r(\cos\theta + i\sin\theta)$ has exactly n complex n^{th} roots (Here, n is a positive integer and r is positive.) The roots are $\sqrt[n]{r}(\cos\phi + i\sin\phi)$, where $\phi = \frac{\theta}{n}, \frac{\theta+360°}{n}, \frac{\theta+720°}{n}, \ldots, \frac{\theta+(n-1)360°}{n}$.
  - The n complex roots of $r(\cos\theta + i\sin\theta)$ are evenly spaced on the circle of radius $\sqrt[n]{r}$ centered at the origin.
  - The easiest way to find the n^{th} roots of any complex number $a + bi$ is to convert it to trigonometric form and use this method.